BER Sensitivity of OFDM Systems to Carrier Frequency Offset and Wiener Phase Noise

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Abstract—In this contribution the transmission of M-PSK and M-QAM modulated orthogonal frequency division multiplexed (OFDM) signals over an additive white Gaussian noise (AWGN) channel is considered. The degradation of the bit error rate (BER), caused by the presence of carrier frequency offset and carrier phase noise is analytically evaluated. It is shown that for a given BER degradation, the value of the frequency offset and the linewidth of the carrier generator that are allowed for OFDM are orders of magnitude smaller than for single carrier systems carrying the same bit rate.

Keywords—OFDM, Wiener noise, carrier frequency offset, BER degradation

I. BIT ERROR RATE DEGRADATION FOR OFDM SIGNALS

An OFDM signal consists of N sinusoids (bins) with spacing 1/T, that are modulated by data symbols that have a duration T, equal to the inverse channel spacing. Hence, although the modulated bins spectrally overlap, they are orthogonal. The generation of the OFDM signals at the transmitter and the demodulation at the receiver can be performed efficiently by means of Fast Fourier Transforms. The use of OFDM signals has been proposed for various applications, such as broadcasting of digital audio and digital TV, and high bit rate transmission on twisted pair cables [1-4].

During a symbol period T, the complex envelope of the transmitted OFDM signal can be expressed as

\[ s(t) = \sum_{m=0}^{N-1} a_m e^{j2\pi m t/T} e^{j\theta(t)}. \]

The data symbol \( a_m \) modulates the sinusoid at frequency \( m/T \) during a period T. The total symbol rate R equals N/T. In the following \( \theta(t) \) denotes a time varying phase caused by either a carrier offset between the receiver and transmitter carrier, or the phase noise of these carriers. In the first case, \( \theta(t) \) is deterministic and equals \( 2\pi\Delta f t + \phi_0 \) where \( \Delta f \) is the carrier offset. In the latter case, \( \theta(t) \) is modeled as a Wiener process for which \( \mathbb{E}[\theta(t)] = 0 \) and \( \mathbb{E}[(\theta(t + \tau) - \theta(t))^2] = 4\pi^2 \beta^2 |\tau| \), where \( \beta [\text{Hz}] \) denotes the one-sided 3 dB linewidth of the Lorentzian power density spectrum of the free-running carrier generator.

During a symbol period, the signal at the FFT output corresponding to the k-th bin, can be written as

\[ r_k = a_k I_0 + \sum_{m=0}^{N-1} a_m I_{k-m} + N_k \] (1)

where

\[ I_n = \frac{1}{T} \int_0^T e^{-j2\pi n t} e^{j\theta(t)} dt. \]

We observe that the presence of phase noise and/or frequency offset affects the received useful signal. Both impairments rotate and attenuate the useful symbol \( a_k \). In addition, signal components originating from bins other than the considered one give rise to interbin interference (IBI). The third term in (1) denotes the thermal noise contribution of the AWGN channel.

The receiver rotates clockwise the signal \( r_k \) over an angle \( \Psi \) which is an estimate of \( \arg(I_0) \), and feeds the resulting signal \( v_k \) to the decision device. In the following, we (optimistically) assume that the receiver can perfectly estimate \( \arg(I_0) \), i.e. \( \Psi = \arg(I_0) \). This yields

\[ v_k = r_k e^{-j\Psi} = a_k I_0 + \left( \sum_{m=0}^{N-1} a_m I_{k-m} + N_k \right) e^{-j\Psi}. \]

Writing \( |I_0| \) as \( E_0 + \delta \) with \( E_0 = E[|I_0|] \), the useful signal component in \( v_k \) is \( a_k E_0 \) whereas \( a_k \delta \) is an additional noise component.

The signal-to-noise-ratio (SNR) at the input of the decision device is given by

\[ \text{SNR} = \frac{E_0^2}{N_0 + V_0}. \] (2)

where \( E_0^2 \) is the power of the useful component of \( v_k \), \( N_0 / E_s \) denotes the variance of the thermal noise contribution, and \( V_0 \) equals the variance of the other noise terms, i.e.

\[ V_0 = E[\delta^2] + \sum_{m=0}^{N-1} E[|I_{k-m}|^2]. \] (3)

Comparing (2) with SNR = \( E_s / N_0 \) in absence of carrier phase impairments, we define the degradation in dB as

\[ D = -10\log\left( \frac{E_0^2}{N_0} \right) \]

\[ = -10\log E_0^2 + 10\log(1 + V_0 E_s N_0). \] (4)

The first term in (4) is caused by the reduction of the useful signal amplitude \( E_0 < 1 \), whereas the second term in (4) is due to the extra noise term (consisting of \( \delta \) and IBI) having a variance \( V_0 \). When these impairments are small, D is well approximated by

\[ D \approx 10 \ln 10 \left( (1 - E_0^2) + V_0 E_s N_0 \right). \] (5)
For small values of $D$, $E_s/N_0$ has to be increased by an amount essentially equal to $D$ (in dB) in order to maintain the same BER as in the case where $I_0 = 1$ and $I_n = 0$ for $n \neq 0$, i.e. in the absence of carrier phase impairments; this can be verified by a similar reasoning as in [5]. Therefore, $D$ can be interpreted as the BER degradation.

Defining $E_1 = E[I_0^2]$, the two components of $V_0$ in (3) can be manipulated as follows:

$$E[|\delta|^2] = E_1 - E_0^2,$$

$$\sum_{n=1}^{N-1} E[|I_{k-n}|^2] = 1 - E_1.$$

This last expression is valid for a large number ($N$) of bins (see appendix). Hence,

$$V_0 = (E_1 - E_0^2) + (1 - E_1).$$

In the case of single carrier (SC) systems with a rectangular baseband pulse of duration $1/R$, $V_0$ contains only the first term of (6) because there is obviously no IBI. When the fluctuation of $|I_0|^2$ with respect to $E_1$ is small, we obtain

$$E_0^2 = (E[E_1 + (|I_0|^2 - E_1)]^{1/2})^2 
\approx E_1 - V_1 = 4E_1.$$

where

$$V_1 = E[|I_0|^4] - E_1^2.$$
Fig. 2. Distortion as function of the oscillator linewidth

APPENDIX

As \( E[|I_m|^2] \) decreases with increasing \( |m| \), the variance of the IBI which is present at the \( k \)-th bin is approximated by

\[
\sum_{m=-N}^{N} E[|I_{k-m}|^2] \approx \sum_{m=-\infty}^{+\infty} E[|I_{m}|^2] - E_1, \tag{A.1}
\]

which assumes an infinite number of bins to the left and right of the considered \( k \)-th bin; for \( k=0 \) and \( k = N-1 \), the above approximation overestimates the IBI variance by a factor of 2, but is very accurate for bins that are not close to the band edges.

The first term in (A.1) can be transformed into

\[
\sum_{m=-\infty}^{+\infty} E[|I_{m}|^2] = \tag{A.2}
\]

\[
\frac{1}{T^2} \int_0^T \int_0^T E[e^{j(\theta(u) - \theta(v))}] \sum_{m=-\infty}^{+\infty} e^{-j2\pi (u-v)m}du dv .
\]

Using in (A.2) the identity

\[
\sum_{m=-\infty}^{+\infty} e^{-j2\pi t m} = T \sum_{m=-\infty}^{+\infty} \delta(t - mT)
\]

yields

\[
\sum_{m=-\infty}^{+\infty} E[|I_{m}|^2] = 1 .
\]

REFERENCES